

The inertia set of a signed graph

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Abstract

A signed graph is a pair (G, Σ) , where $G = (V, E)$ is a graph (in which parallel edges are permitted, but loops are not) with $V = \{1, \dots, n\}$ and $\Sigma \subseteq E$. By $S(G, \Sigma)$ we denote the set of all symmetric $V \times V$ matrices $A = [a_{i,j}]$ with $a_{i,j} < 0$ if i and j are connected by only even edges, $a_{i,j} > 0$ if i and j are connected by only odd edges, $a_{i,j} \in \mathbb{R}$ if i and j are connected by both even and odd edges, $a_{i,j} = 0$ if $i \neq j$ and i and j are non-adjacent, and $a_{i,i} \in \mathbb{R}$ for all vertices i . The stable inertia set of a signed graph (G, Σ) is the set of all pairs (p, q) for which there exists a matrix $A \in S(G, \Sigma)$ with p positive and q negative eigenvalues which has the Strong Arnold Property. In this paper, we study the stable inertia set of (signed) graphs.

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1 Introduction

A *signed graph* is a pair (G, Σ) , where $G = (V, E)$ is a graph (in which parallel edges are permitted, but loops are not) with $V = \{1, \dots, n\}$ and $\Sigma \subseteq E$. (We refer to [7] for the notions and concepts in Graph Theory.) The edges in Σ are called *odd* and the other edges of E *even*. If $V = \{1, 2, \dots, n\}$, we denote by $S(G, \Sigma)$ the set of all real symmetric $n \times n$ matrices $A = [a_{i,j}]$ with

- $a_{i,j} < 0$ if i and j are connected by only even edges,
- $a_{i,j} > 0$ if i and j are connected by only odd edges,
- $a_{i,j} \in \mathbb{R}$ if i and j are connected by both even and odd edges,
- $a_{i,j} = 0$ if $i \neq j$ and i and j are non-adjacent, and
- $a_{i,i} \in \mathbb{R}$ for all vertices i .

If A is a symmetric matrix, then by $\text{pin}(A)$ we denote the pair (p, q) , where p and q are the number of positive and negative eigenvalues of A , respectively. We define the *inertia set* of a signed graph (G, Σ) as the set $\{\text{pin}(A) \mid A \in S(G, \Sigma)\}$ and denote it by $\mathcal{I}(G, \Sigma)$. The analogous version for graphs was introduced by Barrett, Hall, and Loewy in [2]. For a graph G , denote by $S(G)$ the set of all real symmetric $n \times n$ matrices $A = [a_{i,j}]$ with $a_{i,j} \neq 0$ if i and j are connected by a single edge, $a_{i,j} \in \mathbb{R}$ if i and j are connected by multiple edges, $a_{i,j} = 0$ if $i \neq j$ and i and j are non-adjacent, and $a_{i,i} \in \mathbb{R}$ for all vertices i . The inertia set of a graph G is the set $\{\text{pin}(A) \mid A \in S(G)\}$ and is denoted by $\mathcal{I}(G)$. The inertia set of a signed graph is a refinement of the inertia set of a graph: If $G = (V, E)$ is a graph, then $\mathcal{I}(G) = \cup_{\Sigma \subseteq E} \mathcal{I}(G, \Sigma)$.

We also make the following definitions for signed graphs. The *minimum rank* of a signed graph (G, Σ) , denoted $\text{mr}(G, \Sigma)$, is the minimum of the ranks of the matrices in $S(G, \Sigma)$. The *minimum semidefinite rank* of a signed graph (G, Σ) , denoted $\text{mr}_+(G, \Sigma)$, is the minimum of the ranks of the positive semidefinite matrices in $S(G, \Sigma)$. The *maximum nullity* of a signed graph (G, Σ) , denoted $M(G, \Sigma)$, is the maximum of the nullities of the matrices in $S(G, \Sigma)$, and the *maximum semidefinite nullity*, denoted $M_+(G, \Sigma)$, is the maximum of the nullities of the positive semidefinite matrices in $S(G, \Sigma)$. It is clear that if (G, Σ) has n vertices, then $M(G, \Sigma) + \text{mr}(G, \Sigma) = n$ and $M_+(G, \Sigma) + \text{mr}_+(G, \Sigma) = n$. The minimum rank, minimum semidefinite rank, the maximum nullity, and the maximum semidefinite nullity of a graph G are defined in the same way as the corresponding parameters for signed graphs, except that one replaces $S(G, \Sigma)$ by $S(G)$. Clearly, for a graph $G = (V, E)$, $\text{mr}(G) = \min_{\Sigma \subseteq E} \text{mr}(G, \Sigma)$ and $\text{mr}_+(G) = \min_{\Sigma \subseteq E} \text{mr}_+(G, \Sigma)$. The inertia set generalizes the minimum rank and minimum semidefinite rank. The minimum rank (minimum semidefinite rank) is equal to the smallest integer $k \geq 0$ such that there exists a pair $(p, q) \in \mathcal{I}(G, \Sigma)$ with $p + q = k$ (such that there exists a pair $(p, 0) \in \mathcal{I}(G, \Sigma)$ with $p = k$). Part of the results in this paper are characterizations of the classes of signed graphs (G, Σ) with $M_+(G, \Sigma) \leq 1$ and with $M(G, \Sigma) \leq 1$. Currently, the study of minimum rank and minimum semidefinite rank of a graph is an area of active

research. For a survey on the minimum and minimum semidefinite ranks of graphs, we refer to [8].

For an integer $n \geq 0$, let $\mathbb{N}_{[0,n]}^2$ denote the set of all pair $(p, q) \in \mathbb{N}^2$ with $p+q \leq n$. (Here we include 0 in the set of all natural numbers \mathbb{N} .) To obtain the inertia set of a signed graph (G, Σ) with n vertices, we need to check for each $(p, q) \in \mathbb{N}_{[0,n]}^2$ whether or not there exists a matrix $A \in S(G, \Sigma)$ with p positive and q negative eigenvalues. Some pairs (p, q) for which there exists such a matrix $A \in S(G, \Sigma)$ can be found using the stable inertia set (defined below). In general, using the stable inertia set we do not find all pairs of the inertia set of a signed graph, but we will see that in many cases the stable inertia set is already sufficient to determine the inertia set of a signed graph, and if not, we can at least partially determine the inertia set of a signed graph. Before describing what we mean by the stable inertia set of a signed graph, we describe graph parameters that are special cases of the stable inertia set.

Colin de Verdière introduced in [4] the interesting graph parameter μ . In order to describe this parameter we need the notion of *Strong Arnold Property* (SAP for short). A matrix $A = [a_{i,j}] \in S(G)$ has the SAP if $X = 0$ is the only symmetric matrix $X = [x_{i,j}]$ such that $x_{i,j} = 0$ if i and j are adjacent vertices or $i = j$, and $AX = 0$. For a graph G , $\mu(G)$ is defined as the largest nullity of any matrix $A \in S(G, \emptyset)$ that has exactly one negative eigenvalue and has the SAP. This parameter has a very nice property: if H is a minor of G , then $\mu(H) \leq \mu(G)$. Recall that a minor of a graph G is any graph that can be obtained from G by a series of contractions of edges of subgraph of G . Further, the parameter μ characterizes outerplanar and planar graphs as those graphs G such that $\mu(G) \leq 2$ and $\mu(G) \leq 3$, respectively, see [4]. Lovász and Schrijver [17] showed that graphs G that have a flat embedding are exactly those for which $\mu(G) \leq 4$. For a survey on μ , see [15].

In [6], Colin de Verdière introduced the graph parameter ν . For a simple graph G , $\nu(G)$ is defined to be the largest nullity of any positive semidefinite matrix $A \in S(G)$ having the SAP. This parameter also has the very nice property that if H is a minor of G , then $\nu(H) \leq \nu(G)$. (As $\nu(G)$ is a lower bound of $M_+(G)$, it can be used to obtain lower bounds for $M_+(G)$ using minors of G .) For the complete graph K_n with $n > 1$, $\nu(K_n) = n - 1$. So if a simple graph G contains a cycle, that is, if G contains a minor isomorphic to K_3 , then $\nu(K_3) = 2 \leq M_+(G)$. Hence a simple graph G with $M_+(G) \leq 1$ does not contain any cycle, that is, G is a forest. Colin de Verdière showed in [6] that for simple graphs, $\nu(G) \leq 1$ if and only if G is a forest. The simple graphs G with $\nu(G) \leq 2$ have been characterized by Kotlov [16]. The parameter ν can be extended to graphs in which we allow parallel edges, but no loops; see [12]. In [13], van der Holst gave a characterization of graphs G with $\nu(G) \leq 2$, where parallel edges are permitted.

Barioli, Fallat, and Hogben introduced in [1] a similar graph parameter ξ for simple graphs G . This parameter is defined as the largest nullity of any matrix $A \in S(G)$ having the SAP. It has the same property that if H is a minor of G , then $\xi(H) \leq \xi(G)$. For complete graphs K_n with $n > 1$, $\xi(K_n) = n - 1$. Furthermore, $\xi(K_{1,3}) = 2$. So if a simple graph G contains a cycle or a vertex with degree > 2 , then $2 \leq \xi(G) \leq M(G)$. Only a disjoint union of paths satisfy the conditions of having no cycles and no vertices with degree > 2 . Thus, if $\xi(G) \leq 1$, then G is a disjoint union of paths.

Barioli, Fallat, and Hogben showed in [1] that $\xi(G) \leq 1$ if and only if G is a disjoint union of paths. The graphs G with $\xi(G) \leq 2$ also have been characterized, see [11].

We define the *stable inertia set* of a signed graph (G, Σ) as the set

$$\{\text{pin}(A) \mid A \in S(G, \Sigma) \text{ and } A \text{ has the SAP}\},$$

and denote it by $\mathcal{I}^s(G, \Sigma)$. It is clear that $\mathcal{I}^s(G, \Sigma) \subseteq \mathcal{I}(G, \Sigma)$ for any signed graph (G, Σ) . Analogously, we define the stable inertia set of a graph $G = (V, E)$ as the set

$$\{\text{pin}(A) \mid A \in S(G) \text{ and } A \text{ has the SAP}\},$$

and denote it by $\mathcal{I}^s(G)$. It is easy to see that $\mathcal{I}^s(G) = \cup_{\Sigma \subseteq E} \mathcal{I}^s(G, \Sigma)$ and that $\mathcal{I}^s(G) \subseteq \mathcal{I}(G)$.

If v is a vertex of (G, Σ) , then $\delta(v)$ denotes the set of all edges incident with v . We call the operation $\Sigma \rightarrow \Sigma \Delta \delta(v)$ *resigning around v* , where Δ is the symmetric difference. If $U \subseteq V$, then $\delta(U)$ denotes the set of all edges that have one end in U and one end not in U . We say that (G, Σ) and $(G, \Sigma \Delta \delta(U))$ are *sign-equivalent* and call the operation $\Sigma \rightarrow \Sigma \Delta \delta(U)$ *resigning on U* . Resigning on U amounts to performing a diagonal similarity on the matrices in $S(G, \Sigma)$, and hence it does not affect the inertia set. We call a cycle C of a signed graph (G, Σ) *odd* if C has an odd number of odd edges, otherwise we call C *even*. We call a signed graph *bipartite* if it has no odd cycles. Zaslavsky showed in [18] that two signed graphs are sign-equivalent if and only if they have the same set of odd cycles. Thus, signed graphs that have the same set of odd cycles have the same (stable) inertia set (and the same minimum rank and the same minimum semidefinite rank).

In Section 4, the main section of the paper, we will show that the stable inertia set of a signed graph behaves well under taking subgraphs and contracting edges. (Contracting an edge e with ends u and v means deleting e and identifying u and v , and since we do not allow loops, also deleting any loops that appear.) From this result it follows that the same holds for the stable inertia set of a graph. The stable inertia set of a signed graph includes in a simple way the graph parameters μ, ν , and ξ . For a graph $G = (V, E)$ with n vertices, $\mu(G)$ is the largest integer $k \geq 0$ such that $(n - k - 1, 1) \in \mathcal{I}^s(G, \emptyset)$, $\nu(G)$ is the largest integer $k \geq 0$ such that $(n - k, 0) \in \mathcal{I}^s(G)$, and $\xi(G)$ is the largest integer $k \geq 0$ for which there exists an integer $p \geq 0$ such that $(p, n - k - p) \in \mathcal{I}^s(G)$.

For a signed graph (G, Σ) , we define $\nu(G, \Sigma)$ as the largest nullity of any positive semidefinite matrix $A \in S(G, \Sigma)$ having the SAP, and $\xi(G, \Sigma)$ as the largest nullity of any matrix $A \in S(G, \Sigma)$ having the SAP. If (G, Σ) has n vertices, then $\nu(G, \Sigma)$ is the largest integer $k \geq 0$ such that $(n - k, 0) \in \mathcal{I}^s(G, \Sigma)$ and $\xi(G, \Sigma)$ is the largest integer $k \geq 0$ for which there exists an integer $p \geq 0$ such that $(p, n - k - p) \in \mathcal{I}^s(G, \Sigma)$. A *minor* of a signed graph (G, Σ) is any signed graph that can be obtained from (G, Σ) by deleting edges and vertices, contracting even edges, and resigning around vertices. From the result on the behavior of the stable inertia set of a signed graph under taking subgraphs and contracting edges, it follows that if (H, Ω) is a minor of (G, Σ) , then $\nu(H, \Omega) \leq \nu(G, \Sigma)$, and that if H' is a minor of G , then $\xi(H', \Sigma \cap E(H')) \leq \xi(G, \Sigma)$.

Let us now introduce some notation. Let $A = [a_{i,j}]$ be an $n \times n$ matrix. If $R, S \subseteq \{1, \dots, n\}$, we denote by $A[R, S]$ the submatrix of A that lies in the rows of A indexed

by R and the column indexed by S . If $R = S$, we will write $A[S]$ for $A[R, S]$; $A[\{v\}, S]$ can be denoted by $A[v, S]$. If $S \subseteq \{1, \dots, n\}$, we let $\bar{S} = \{1, \dots, n\} \setminus S$. If $A[S]$ is invertible, the Schur complement of $A[S]$ in A is the matrix

$$A/A[S] = A[\bar{S}] - A[\bar{S}, S]A[S]^{-1}A[S, \bar{S}].$$

Notice that if $A[S]$ is invertible, then $\det(A) = \det(A[S]) \det(A/A[S])$. If m and n are nonnegative integers, we denote the space of all $m \times n$ real matrices by $M_{m,n}$. If n is a nonnegative integer, we denote the set of all symmetric $n \times n$ real matrices by S_n .

2 Stable Northeast Lemma for signed graphs

In this section, we show that if $(p, q) \in \mathcal{I}^s(G, \Sigma)$ and $p + q < |V(G)|$, then the pair up (to the north) and the pair to the right (to the east) also belong to $\mathcal{I}^s(G, \Sigma)$. This theorem was shown for the inertia sets of graphs by Barrett et al. [2].

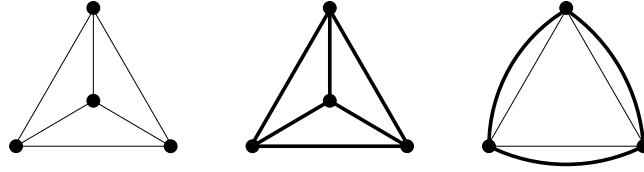


Figure 1: K_4^e , K_4^o , and $K_3^=$

Let us first introduce some signed graphs. By K_n^e and K_n^o we denote the signed graphs (K_n, \emptyset) and $(K_n, E(K_n))$, respectively. By $K_n^=$, we denote the signed graph (G, Σ) , where G is the graph obtained from K_n by adding to each edge an edge in parallel, and where Σ is the set of edges of K_n . By C_n^e we denote the signed graph (C_n, \emptyset) , and by C_n^o we denote the signed graph $(C_n, \{e\})$, where e is an edge of C_n . In Figure 1, we have depicted K_4^e , K_4^o , and $K_3^=$. Here a bold edge is an odd edge. By K_4^d , we denote the signed graph $(K_4, \{e\})$, where e is an edge of K_4 . By $K_{2,3}^e$ and $K_{2,3}^o$, we denote the signed graphs $(K_{2,3}, \emptyset)$ and $(K_{2,3}, \{e\})$, where e is an edge of $K_{2,3}$, respectively.

We have the following observation: if $(p, q) \in \mathcal{I}^s(G, \Sigma)$, then $(q, p) \in \mathcal{I}^s(G, E(G) \setminus \Sigma)$. A similar statement holds for $\mathcal{I}(G, \Sigma)$. Hence, if (G, Σ) is isomorphic to a signed graph that is sign-equivalent to $(G, E(G) \setminus \Sigma)$, then $(q, p) \in \mathcal{I}^s(G, \Sigma)$ if $(p, q) \in \mathcal{I}^s(G, \Sigma)$, and, of course, a similar statement holds for $\mathcal{I}(G, \Sigma)$. Among the signed graphs that we introduced above, $K_n^=$, $K_{2,3}^e$, $K_{2,3}^o$, K_4^d , C_n^e , and C_n^o , with n even, have this property.

Lemma 1 (Stable Northeast Lemma for Signed Graphs). *Let (G, Σ) be a signed graph with n vertices. If $(p, q) \in \mathcal{I}^s(G, \Sigma)$ and $p + q < n$, then $(p + 1, q), (p, q + 1) \in \mathcal{I}^s(G, \Sigma)$.*

Proof. Let $A \in S(G, \Sigma)$ have the SAP, p positive and q negative eigenvalues, and let $k = p + q$. There is a $k \times k$ diagonal matrix D with p positive and q negative numbers on the diagonal, and a $k \times n$ matrix U with $\text{rank}(U) = k$ such that $A = U^T D U$. Denote by e_i the vector whose i th coordinate is 1 and all other coordinates are 0. Since $k < n$, there is an $1 \leq i \leq n$ such that

$$\begin{bmatrix} U \\ e_i^T \end{bmatrix}$$

has rank $k + 1$. Let

$$B = A + e_i e_i^T.$$

Since $A \in S(G, \Sigma)$, clearly $B \in S(G, \Sigma)$. Since

$$B = \begin{bmatrix} U^T & e_i \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U \\ e_i^T \end{bmatrix},$$

B has, by Sylvester's Law of Inertia, $p + 1$ positive and q negative eigenvalues.

It remains to show that B has the SAP. The null space of B is equal to the null space of $\begin{bmatrix} U \\ e_i^T \end{bmatrix}$, so the null space of B is a subspace of the null space of A . Let $X = [x_{i,j}]$ be a symmetric $n \times n$ matrix with $x_{i,j} = 0$ if $i = j$ or $ij \in E$ such that $BX = 0$. Since each vector in the null space of B is in the null space of A , we see that $AX = 0$. As A has the SAP, $X = 0$. Therefore B has the SAP. Thus $(p + 1, q) \in \mathcal{I}^s(G, \Sigma)$. By taking $B = A - e_i e_i^T$, we see that $(p, q + 1) \in \mathcal{I}^s(G, \Sigma)$. \square

Corollary 2 (Stable Northeast Lemma for Graphs). *Let G be a graph with n vertices. If $(p, q) \in \mathcal{I}^s(G)$ and $p + q < n$, then $(p + 1, q), (p, q + 1) \in \mathcal{I}^s(G)$.*

The proof of the next lemma is similar to the one for Lemma 1.

Lemma 3 (Northeast Lemma for Signed Graphs). *Let (G, Σ) be a signed graph with n vertices. If $(p, q) \in \mathcal{I}(G, \Sigma)$ and $p + q < n$, then $(p + 1, q), (p, q + 1) \in \mathcal{I}(G, \Sigma)$.*

As an illustration of Lemma 1, let us determine the inertia sets and the stable inertia sets of the signed graphs $K_n^=$, K_n^e , and K_n^o . We will determine the inertia sets and stable inertia sets of C_n^o , C_n^e , $K_{2,3}^e$, $K_{2,3}^o$, and K_4^d in Section 5.

Proposition 4. $\mathcal{I}(K_n^=) = \mathcal{I}^s(K_n^=) = \mathbb{N}_{[0,n]}^2$.

Proof. The $n \times n$ all-zero matrix belongs to $S(K_n^=)$, has 0 positive and negative eigenvalues, and has the SAP. Hence $(0, 0) \in \mathcal{I}^s(K_n^=)$. By Lemma 1, $\mathbb{N}_{[0,n]}^2 \subseteq \mathcal{I}^s(K_n^=)$. Since clearly $\mathcal{I}(K_n^=) \subseteq \mathbb{N}_{[0,n]}^2$, we obtain $\mathcal{I}(K_n^=) = \mathcal{I}^s(K_n^=) = \mathbb{N}_{[0,n]}^2$. \square

To determine the inertia sets of K_n^e and K_n^o , we first need a lemma which says that the nullity of a connected bipartite signed graph is at most 1.

Lemma 5. *Let (G, Σ) be a connected bipartite signed graph and let $A \in S(G, \Sigma)$ be positive semidefinite. If $x \in \ker(A)$ is nonzero, then x has only nonzero entries. Especially, $\text{nullity}(A) \leq 1$.*

Proof. Since (G, Σ) is bipartite, we can resign the signed graph to (G, Σ') which has only even edges. Resigning around a vertex v corresponds to multiplying the v th row and column of A by -1 , and to multiplying the v entry of x by -1 . Hence the resulting matrix $B \in S(G, \Sigma')$ has only nonpositive off-diagonal entries and the resulting vector y belongs to $\ker(B)$. If a sufficiently large, then $-B + aI$ has only nonnegative entries. Notice that $(-B + aI)y = ay$. Since a is the largest eigenvalue and $-B + aI$ is irreducible, Perron-Frobenius tells us that any nonzero vector in $\ker(-B + aI)$ has all components positive or all components negative. Hence y has only nonzero components, and so has x . If $\text{nullity}(A) > 1$, then there exists a nonzero $x \in \ker(A)$ with at least one component equal to zero. This contradicts that x can have only nonzero components. \square

Lemma 6. *If (G, Σ) is a connected bipartite signed graph with n vertices, then $(n - 2, 0) \notin \mathcal{I}(G, \Sigma)$ and $(n - 1, 0) \in \mathcal{I}^s(G, \Sigma)$.*

Proof. By Lemma 5, $(n - 2, 0) \notin \mathcal{I}(G, \Sigma)$.

To see that $(n - 1, 0) \in \mathcal{I}^s(G, \Sigma)$, let $A \in S(G, \Sigma)$ be positive semidefinite. If the nullity of A is not equal to one, let λ_1 be the smallest eigenvalue of A . Then $A - \lambda_1 I$ is positive semidefinite and has nullity 1. Hence we may assume that A has nullity 1. Any matrix $A \in S(G, \Sigma)$ with nullity 1 has the SAP. To see this, let $X = [x_{i,j}]$ be a symmetric matrix satisfying $AX = 0$ and $x_{i,j} = 0$ if i and j are adjacent or if $i = j$. Then $X = yy^T$ for some $y \in \ker(A)$ and since $x_{i,i} = 0$ for all vertices i , $y = 0$. We can therefore conclude that $(n - 1, 0) \in \mathcal{I}^s(G, \Sigma)$. \square

Lemma 7. *Let (G, Σ) be a signed graph with n vertices. If $S \subseteq V(G)$ and $(G, \Sigma) \setminus S$ is a connected bipartite signed graph, then $(n - 2 - |S|, 0) \notin \mathcal{I}(G, \Sigma)$.*

Proof. Suppose for a contradiction that $(n - 2 - |S|, 0) \in \mathcal{I}(G, \Sigma)$. Let $A \in S(G, \Sigma)$ be a positive semidefinite matrix with nullity $|S| + 2$. Then $A[V - S]$ has nullity at least 2. Since $(G, \Sigma) \setminus S$ is a connected bipartite signed graph, we obtain a contradiction by Lemma 5. \square

For $A, B \subseteq \mathbb{N}^2$, define

$$A + B = \{(p_1 + p_2, q_1 + q_2) \mid (p_1, q_1) \in A, (p_2, q_2) \in B\}.$$

Proposition 8. $\mathcal{I}^s(K_1) = \mathcal{I}(K_1) = \mathbb{N}_{[0,1]}^2$. If $n > 1$, then

$$\mathcal{I}^s(K_n^e) = \mathcal{I}(K_n^e) = [\{(0, 1)\} + \mathbb{N}_{[0, n-1]}^2] \cup \{(n - 1, 0), (n, 0)\}$$

and

$$\mathcal{I}^s(K_n^o) = \mathcal{I}(K_n^o) = [\{(1, 0)\} + \mathbb{N}_{[0, n-1]}^2] \cup \{(0, n - 1), (0, n)\}.$$

Proof. The $n \times n$ all -1 matrix has one negative, no positive eigenvalues, and has the SAP. By the Stable Northeast Lemma, $\{(0, 1)\} + \mathbb{N}_{[0, n-1]}^2 \subseteq \mathcal{I}^s(K_n^e)$. Since K_n^e is a connected bipartite signed graph, $(n - 2, 0) \notin \mathcal{I}(K_n^e)$. Since $\mathcal{I}^s(K_n^e) \subseteq \mathcal{I}(K_n^e)$, $\mathcal{I}^s(K_n^e) = \mathcal{I}(K_n^e) = [\{(0, 1)\} + \mathbb{N}_{[0, n-1]}^2] \cup \{(n - 1, 0), (n, 0)\}$. The (stable) inertia set of K_n^o follows immediately from the (stable) inertia set of K_n^e . \square

If $0 \leq k \leq n$, we define $\mathbb{N}_{[k,n]}^2 = \{(p, q) \in \mathbb{N}^2 \mid k \leq p + q \leq n\}$.

In [2], Barrett, Hall, and Loewy showed that $\mathcal{I}(K_n) = \mathbb{N}_{[1,n]}^2$. From Proposition 8, we obtain the following extension of their result.

Corollary 9. $\mathcal{I}^s(K_n) = \mathcal{I}(K_n) = \mathbb{N}_{[1,n]}^2$.

Proof. Since $\mathcal{I}^s(K_n^e) \cup \mathcal{I}^s(K_n^o) \subseteq \mathcal{I}^s(K_n)$ and $\mathbb{N}_{[1,n]}^2 \subseteq \mathcal{I}^s(K_n^e) \cup \mathcal{I}^s(K_n^o)$, $\mathbb{N}_{[1,n]}^2 \subseteq \mathcal{I}^s(K_n)$. Since $\text{mr}(K_n) = 1$, $\mathcal{I}(K_n) \subseteq \mathbb{N}_{[1,n]}^2$. Hence $\mathcal{I}^s(K_n) = \mathcal{I}(K_n) = \mathbb{N}_{[1,n]}^2$. \square

3 The Strong Arnold Hypothesis

Let $G = (V, E)$ be a graph and $A \in S(G)$. Suppose $R \subseteq V$ such that $A[R]$ is invertible and $A/A[R] = 0$. Let U be an open neighborhood around A in $S(G)$ such that $B[R]$ is invertible for each $B \in U$. Define

$$\psi_S : U \rightarrow \mathcal{S}_k$$

by

$$\psi_R(B) = B/B[R], \quad B \in U.$$

If it is clear from the context which set R we use, we will write ψ instead of ψ_R . In this section we show that A has the SAP if and only if the derivative, $D\psi_R(A)$, of ψ_R at A is surjective. First we give an alternative criterion for a matrix $A \in S(G)$ to have the SAP; this is the criterion that Colin de Verdière used in his papers [4, 5, 6].

If G is a graph with vertex set $\{1, \dots, n\}$, we denote by $T(G)$ the set of all symmetric $n \times n$ matrices $B = [b_{i,j}]$ with $b_{i,j} = 0$ if $i \neq j$ and i and j are not adjacent. In the following proof we will use that $\text{Tr}(CD) = \text{Tr}(DC)$ for any $n \times m$ matrix C and any $m \times n$ matrix D .

Theorem 10. *Let G be a graph with n vertices. A matrix $A \in S(G)$ has the SAP if and only if for each real symmetric $n \times n$ matrix N , there is a matrix $B \in T(G)$ such that $x^T N x = x^T B x$ for all $x \in \ker(A)$.*

Proof. Let y_1, \dots, y_k form a basis of $\ker(A)$ and let $Y = [y_1 \dots y_k]$. The statement that for each real symmetric $n \times n$ matrix N , there is a matrix $B \in T(G)$ such that $x^T N x = x^T B x$ for each $x \in \ker(A)$ is the same as the statement that for each real symmetric $n \times n$ matrix N , there is a matrix $B \in T(G)$ such that $Y^T N Y = Y^T B Y$.

Suppose that A does not have the SAP. Then there exists a nonzero real symmetric matrix X with $x_{i,i} = 0$ and $x_{i,j} = 0$ if i and j are adjacent in G , such that $AX = 0$. Since $x_{i,j} = 0$ if i and j are adjacent or if $i = j$, $\text{Tr}(XK) = 0$ for all $K \in T(G)$. Since each column of X belongs to $\ker(A)$, there exists a nonzero symmetric $k \times k$ matrix C such that $X = YCY^T$. Hence $\text{Tr}(YCY^T K) = 0$ for all $K \in T(G)$, and therefore

$$\text{Tr}(CY^T K Y) = 0 \quad \text{for all } K \in T(G).$$

Let N be a symmetric $n \times n$ matrix such that $Y^T N Y = C$. If there would exist a symmetric matrix $B \in T(G)$ such that $C = Y^T B Y$, then $\text{Tr}(Y^T B Y Y^T K Y) = 0$

for all $K \in T(G)$. Especially, $\text{Tr}(Y^T B Y Y^T B Y) = 0$, and hence $C = Y^T B Y = 0$. This contradiction shows that there is no such $B \in T(G)$.

Conversely, suppose that there exists a real symmetric $n \times n$ matrix N such that there is no matrix $B \in T(G)$ for which $x^T N x = x^T B x$ for each $x \in \ker(A)$. Then the space of all matrices $Y^T B Y$ with $B \in T(G)$ is not equal to the space of all real symmetric $k \times k$ matrices. This means that there exists a real symmetric $k \times k$ matrix C such that $\text{Tr}(Y^T B Y C) = 0$ for all $B \in T(G)$. Since $\text{Tr}(Y^T B Y C) = \text{Tr}(B Y C Y^T) = 0$, we see that $Y C Y^T$ is orthogonal to each matrix in $T(G)$. So $X = [x_{i,j}] = Y C Y^T$ is a nonzero matrix with $x_{i,j} = 0$ if $i = j$ or if i and j are adjacent in G for which $A X = 0$. \square

Let L be a subspace of $M_{m,n}$. If $f : L \times L \rightarrow M_{m,n}$ is defined by $f(A, B) = A + B$, then the derivative of f at (A, B) is the linear map $Df(A, B) : L \times L \rightarrow M_{m,n}$ defined by

$$(1) \quad Df(A, B)(C, D) = C + D.$$

Let L_1 and L_2 be subspaces of $M_{m,n}$ and $M_{n,p}$, respectively. If $g : L_1 \times L_2 \rightarrow M_{m,p}$ is defined by $g(A, B) = AB$, then the derivative of g at (A, B) is the linear map $Dg(A, B) : L_1 \times L_2 \rightarrow M_{m,p}$ defined by

$$(2) \quad Dg(A, B)(C, D) = CB + AD.$$

Let L be a subspace of the space of all $n \times n$ matrices and let U be the open set of all invertible matrices in L . If $h : U \rightarrow M_n$ is defined by $h(A) = A^{-1}$, then the derivative of h at A is the linear map $Dh(A) : L \rightarrow M_n$ defined by

$$(3) \quad Dh(A)(C) = -A^{-1} C A^{-1}.$$

Define

$$\gamma_R(A) = \begin{bmatrix} -A[R]^{-1} A[R, \bar{R}] \\ I_k \end{bmatrix}.$$

The columns of $\gamma_R(A)$ form a basis of $\ker(A)$. The mapping ψ_R is C^1 and, by Equations 1, 2, and 3, the derivative at A is the linear map

$$D\psi_R(A) : T(G) \rightarrow S_k$$

defined by

$$D\psi_R(A)(C) = \gamma_R(A)^T C \gamma_R(A).$$

(A C^1 map is a differentiable map whose derivative is continuous.)

Finally, we present our main result of this section; it is a new result that will be important in the next section.

Theorem 11. *Let $A \in S(G)$ and let $R \subseteq V(G)$ such that $A[R]$ is invertible and $\psi_R(A) = 0$. Then A has the SAP if and only if $D\psi_R(A)$ is surjective.*

Proof. Suppose $A \in S(G)$ has the SAP. Let $B \in \mathcal{S}_k$. The symmetric $n \times n$ matrix

$$B' = \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix}$$

satisfies $\gamma_R(A)^T B' \gamma_R(A) = B$. Since A has the SAP, there exists a matrix $C \in T(G)$ such that $\gamma_R(A)^T C \gamma_R(A) = \gamma_R(A)^T B' \gamma_R(A)$. Thus $\gamma_R(A)^T C \gamma_R(A) = B$.

Conversely, suppose $D\psi_R(A)$ is surjective. Let B' be a symmetric $n \times n$ matrix and let $B = \gamma_R(A)^T B' \gamma_R(A)$. There exists a matrix $C \in T(G)$ such that $D\psi_R(A)(C) = \gamma_R(A)^T C \gamma_R(A) = B$. Thus $\gamma_R(A)^T B' \gamma_R(A) = \gamma_R(A)^T C \gamma_R(A)$. \square

Let $A \in S(G, \Sigma)$ and let $R \subseteq V(G)$ such that $A[R]$ is invertible and $\psi_R(A) = 0$. Suppose that $D\psi_R(A)$ is surjective. Since $D\psi_R(A)$ is a continuous function in A , there exists a neighborhood U of A in $S(G, \Sigma)$ such that $D\psi_R(B)$ is surjective for all $B \in U$.

4 The stable inertia set and subgraphs and contracting edges

This is the main section of the paper. If $k \in \mathbb{N}$, then by \mathbb{N}_k^2 we denote the set $\{(p, q) \in \mathbb{N}^2 \mid p + q = k\}$. Let (G, Σ) be a signed graph with n vertices and let G' be a subgraph of G with m vertices. In this section we prove that if (H, Ω) is obtained from $(G', \Sigma \cap E(G'))$ by contracting s even and t odd edges, then

$$\mathcal{I}^s(H, \Omega) + \mathbb{N}_{n-m}^2 + \{(s, t)\} \subseteq \mathcal{I}^s(G, \Sigma).$$

First we state the Implicit Function Theorem and give some lemmas.

Let $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m$ and $W \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be a neighborhood of (x_0, y_0) . If $F : W \rightarrow \mathbb{R}^m$ is differentiable, then $D_1 F(x_0, y_0)$ denotes the derivative of $j(x) = F(x, y_0)$ at $x = x_0$ and $D_2 F(x_0, y_0)$ the derivative of $k(y) = F(x_0, y)$ at $y = y_0$.

Theorem 12 (Implicit Function Theorem). *[10] Let $W \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be an open set and $(x_0, y_0) \in W$. Let $F : W \rightarrow \mathbb{R}^m$ be a C^1 map and $c = F(x_0, y_0)$. Suppose $(x_0, y_0) \in W$ is such that the linear operator $D_2 F(x_0, y_0) : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is invertible. Then there are open sets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ with*

$$(x_0, y_0) \in U \times V \subseteq W$$

and a unique C^1 map $g : U \rightarrow V$ such that

$$F(x, g(x)) = c$$

for all $x \in U$, and moreover, $F(x, y) \neq c$ if $(x, y) \in U \times V$ and $y \neq g(x)$.

We apply the Implicit Function Theorem in the proof of the following lemma.

Lemma 13. *Let $W \subseteq \mathbb{R} \times \mathbb{R}^n$ be an open set and $(0, a) \in W$. Let $F : W \rightarrow \mathbb{R}^m$ be a C^1 map, and define $f(x) = F(0, x)$. If $f(a) = 0$ and $Df(a)$ is surjective, then there exists a $\delta > 0$ such that for each $h \in \mathbb{R}$ with $|h| < \delta$, there exists a $(h, b) \in W$ such that $F(h, b) = 0$ and $DF(h, b)$ is surjective.*

Proof. Since $Df(a)$ is surjective, the Jacobian matrix $Jf(a)$ of f at a has rank m . We may assume that the last m columns of $Jf(a)$ span an invertible matrix. We write $\mathbb{R}^n = \mathbb{R}^{n-m} \times \mathbb{R}^m$ and $a = (a_1, a_2)$, where $a_1 \in \mathbb{R}^{n-m}$ and $a_2 \in \mathbb{R}^m$. Then $D_2f(a_1, a_2)$ is an invertible linear operator, which implies that $DF(0, a)$ is surjective. By the Implicit Function Theorem, there are open sets $U \subseteq \mathbb{R}^{n-m+1}$ and $V \subseteq \mathbb{R}^m$ such that $(0, a_1) \in U$, $a_2 \in V$ and $U \times V \subseteq W$ and a unique C^1 map $G : U \rightarrow V$ such that $F(x, G(x)) = 0$ for all $x \in U$ and $F(x, y) \neq 0$ if $(x, y) \in U \times V$ and $y \neq G(x)$. Let $\delta > 0$ for which there exists a neighborhood U' of a_1 such that $(-\delta, \delta) \times U' \subseteq U$. If $h \in (-\delta, \delta)$, choose $b_1 \in U'$ and let $b = (b_1, G(h, b_1))$. Then $F(h, b) = 0$. \square

The following lemma is taken from [14]; for convenience we include a proof.

Lemma 14. *Let $A = [a_{i,j}]$ be a nonzero symmetric $n \times n$ matrix. Then there is an $S \subseteq \{1, \dots, n\}$ such that $A[S]$ is invertible and $A/A[S] = 0$.*

Proof. Take $S \subseteq \{1, \dots, n\}$ such that $A[S]$ is invertible and $|S|$ is as large as possible. (It is clear that such an S exists.) Let $B = [b_{i,j}] = A/A[S]$. If $b_{i,i} \neq 0$ for some $i \notin S$, then

$$\begin{aligned} \det(A[S \cup \{i\}]) &= \det(A[S]) \det(A[S \cup \{i\}]/A[S]) \\ &= \det(A[S]) b_{i,i} \\ &\neq 0. \end{aligned}$$

This contradicts the maximality of $|S|$. Hence $b_{i,i} = 0$ for each $i \notin S$.

If there are vertices $i, j \notin S$ such that $b_{i,j} \neq 0$, then the principal submatrix

$$K = \begin{bmatrix} 0 & b_{i,j} \\ b_{j,i} & 0 \end{bmatrix}$$

of B has $\det(K) \neq 0$. So

$$\begin{aligned} \det(A[S \cup \{i, j\}]) &= \det(A[S]) \det(A[S \cup \{i, j\}]/A[S]) \\ &= \det(A[S]) \det(K) \\ &\neq 0, \end{aligned}$$

which contradicts the maximality of $|S|$. Thus $B = 0$. \square

Theorem 15. *Let (G, Σ) be a signed graph with n vertices and let $(H, \Sigma \cap E(H))$ be a signed subgraph of (G, Σ) with m vertices. Then*

$$\mathcal{I}^s(H, \Sigma \cap E(H)) + \mathbb{N}_{n-m}^2 \subseteq \mathcal{I}^s(G, \Sigma).$$

Proof. We may assume that (G, Σ) has vertex set $\{1, \dots, n\}$. As each signed subgraph $(H, \Sigma \cap E(H))$ of (G, Σ) arises by a series of deletions of edges and isolated vertices, we may assume that either $H = G \setminus \{v\}$ for an isolated vertex v of G or $H = G \setminus e$ for an edge e .

Assume (H, Σ) arises from (G, Σ) by deleting an isolated vertex v ; we assume that $v = 1$. Let $A = [a_{i,j}] \in S(H, \Sigma)$ have the SAP and have p positive and q negative eigenvalues. The matrix

$$M = \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix}$$

belongs to $S(G, \Sigma)$ and has $p + 1$ positive and q negative eigenvalues. We claim that M has the SAP. Each vector $x \in \ker(M)$ has $x_1 = 0$ and $x[V(H)] \in \ker(A)$. Let C be an arbitrary real symmetric $n \times n$ matrix, $C' = C[V(H)]$, and $y = x[V(H)]$. As A has the SAP, there exists a matrix $B' = [b'_{i,j}] \in T(H)$ such $y^T C' y = y^T B' y$. The matrix

$$B = \begin{bmatrix} 0 & 0 \\ 0 & B' \end{bmatrix}$$

belongs to $T(G)$. We get $x^T C x = y^T C' y = y^T B' y = x^T B x$ for all $x \in \ker(M)$, showing that M has the SAP.

Assume next that $(H, \Sigma \cap E(H))$ arises from (G, Σ) by deleting an even edge $e = uw$; the case where e is odd is similar, except that in that case we have to take $h > 0$. We may assume that $u = 1$ and $w = 2$. Let $R := \{3, \dots, n\}$. Define a function

$$f : \mathbb{R} \times S(H, \Sigma \cap E(H)) \rightarrow \mathcal{S}_n$$

by

$$f(h, K) := \begin{bmatrix} k_{1,1} & h & K[1, R] \\ h & k_{2,2} & K[2, R] \\ K[R, 1] & K[R, 2] & K[R] \end{bmatrix},$$

where $K = [k_{i,j}] \in S(H, \Sigma \cap E(H))$, and let $f_h(K) = f(h, K)$. So f_0 is the identity on $S(H, \Sigma \cap E(H))$.

Let $A = [a_{i,j}] \in S(H, \Sigma \cap E(H))$ have the SAP and have p positive and q negative eigenvalues. By Lemma 14, there exists a subset $S \subseteq V$ for which $A[S]$ is invertible and $\psi_S(A) = 0$. Since A has the SAP, $D\psi_S(A)$ is surjective, by Theorem 11. Since eigenvalues continuously depend on the entries of the matrix, there exists an open set U of $\mathbb{R} \times S(H, \Sigma \cap E(H))$ containing $(0, A)$ such that for all $(h, B) \in U$, $f(h, B)[S]$ has the same inertia as $A[S] = f(0, A)[S]$. Define $g : U \rightarrow \mathcal{S}_k$ by $g = \psi_S \circ f|_U$ and define $g_0(x) = g(0, x)$. Then $g_0(A) = 0$ and $Dg_0(A)$ is surjective. By Lemma 13, there exists a $\delta > 0$ such that for all h_1 with $|h_1| < \delta$, there exists a $C_1 \in S(H, \Sigma \cap E(H))$ such that $(h_1, C_1) \in U$, $g(h_1, C_1) = 0$, and $Dg(h_1, C_1)$ is surjective. Let $\delta < h < 0$ and $C \in S(H, \Sigma \cap E(H))$ such that $(h, C) \in U$, $g(h, C) = 0$ and $Dg(h, C)$ is surjective. Let $M = f(h, C)$. Then $M \in S(G, \Sigma)$. Since $Dg(h, C)$ is surjective, $D\psi_S(M)$ is surjective. Hence M has the SAP. Since $(h, C) \in U$, $M[S]$ has the same inertia as $A[S]$. Thus $\mathcal{I}^s(H, \Sigma \cap E(H)) \subseteq \mathcal{I}^s(G, \Sigma)$. \square

Lemma 16. *If (G, Σ) is a signed graph with $n > 0$ vertices, then $\mathbb{N}_{[n-1, n]}^2 \subseteq \mathcal{I}^s(G, \Sigma) \subseteq \mathcal{I}(G, \Sigma)$.*

Proof. Since $\mathcal{I}^s(K_1) = \mathbb{N}_{[0, 1]}^2$ and by deleting $n - 1$ vertices from (G, Σ) we obtain a single vertex, $\mathbb{N}_{[0, 1]}^2 + \mathbb{N}_{n-1}^2 = \mathbb{N}_{[n-1, n]}^2 \subseteq \mathcal{I}^s(G, \Sigma)$. \square

Barrett et al. [2] showed that for any graph with n vertices, $\mathbb{N}_{[n-1, n]}^2 \subseteq \mathcal{I}(G)$. Lemma 16 is a strengthening of their result, as for any signed graph (G, Σ) , $\mathbb{N}_{[n-1, n]}^2 \subseteq \mathcal{I}^s(G, \Sigma) \subseteq \mathcal{I}^s(G) \subseteq \mathcal{I}(G)$.

If G is a graph without loops and e is an edge of G , then G/e denotes the graph obtained from G by contracting e .

Theorem 17. *Let (G, Σ) be a signed graph and $e \in E(G)$. If e is odd, then*

$$\mathcal{I}^s(G/e, \Sigma \setminus \{e\}) + \{(0, 1)\} \subseteq \mathcal{I}^s(G, \Sigma).$$

If e is even, then

$$\mathcal{I}^s(G/e, \Sigma) + \{(1, 0)\} \subseteq \mathcal{I}^s(G, \Sigma).$$

Proof. We may assume that (G, Σ) has vertex set $\{1, \dots, n\}$.

We give the proof only for the case that $e = uv$ is an even edge; the case that e is odd is similar, except that $h < 0$ when e is odd. We may assume that $u = n - 1$ and $w = n$. Let v be the new vertex of G/e ; so $v = n - 1$.

Let $R = \{1, \dots, n - 2\}$. Let \mathcal{Z} be the subset of all matrices $K = [k_{i,j}] \in S(G, \Sigma)$ with $k_{n,n} = 0 = k_{n,n-1}$. Define a function

$$(4) \quad f : \mathbb{R} \times \mathcal{Z} \longrightarrow \mathcal{S}_{n-1},$$

by

$$(5) \quad f(h, K) = \begin{bmatrix} K[R] - hK[R, n]K[n, R] & K[R, n] + K[R, n - 1] \\ K[n, R] + K[n - 1, R] & k_{n-1, n-1} \end{bmatrix},$$

and let $f_h(K) = f(h, K)$.

Let $A = [a_{i,j}] \in S(G/e, \Sigma)$ have the SAP and have p positive and q negative eigenvalues. By Lemma 14, there exists a subset $S \subseteq \{1, \dots, n - 1\}$ for which $A[S]$ is invertible and $\psi_S(A) = 0$. Since A has the SAP, $D\psi_S(A)$ is surjective by Theorem 11. The image of f_0 is a superset of $S(G/e, \Sigma)$, so there is a $P \in \mathbb{R} \times \mathcal{Z}$ such that $f_0(P) = A$. Since eigenvalues continuously depend on the entries of the matrix, there exists an open set U of $\mathbb{R} \times \mathcal{Z}$ containing $(0, P)$ such that for all $(h, B) \in U$, $f(h, B)[S]$ has the same inertia as $A[S] = f(0, P)[S]$. Define $g : U \rightarrow \mathcal{S}_k$ by $g = \psi_S \circ f|_U$ and define $g_0(x) = g(0, x)$. Then $g_0(P) = 0$. Since $T(G/e)$ is a subset of the image of $Df|_U$, $Dg_0(P)$ is surjective. By Lemma 13, there exists a $\delta > 0$ such that for all h_1 with $|h_1| < \delta$, there exists a $C_1 \in S(G/e, \Sigma)$ such that $(h_1, C_1) \in U$, $g(h_1, C_1) = 0$, and $Dg(h_1, C_1)$ is surjective. Let $0 < h < \delta$ and $C \in S(G/e, \Sigma)$ such that $(h, C) \in U$, $g(h, C) = 0$, and $Dg(h, C)$ is surjective. Let

$$M = \begin{bmatrix} C[R] & C[R, n - 1] & C[R, n] \\ C[n - 1, R] & c_{n-1, n-1} + \frac{1}{h} & -\frac{1}{h} \\ C[n, R] & -\frac{1}{h} & \frac{1}{h} \end{bmatrix}.$$

Then $\psi_{\{n\}}(M) = f(h, C)$, so $g(h, C) = \psi_S(\psi_{\{n\}}(M))$. Since $(h, C) \in U$, $f(h, C)[S]$ has the same inertia as $A[S] = f(0, P)[S]$. Hence M has $p + 1$ positive and q negative eigenvalues. Since $Dg(h, C)$ is surjective, $D(\psi_S \circ \psi_{\{n\}})(M) = D\psi_{S \cup \{n\}}(M)$ is surjective. Hence M has the SAP. Thus $\mathcal{I}^s(G/e, \Sigma) + \{(1, 0)\} \subseteq \mathcal{I}^s(G, \Sigma)$. \square

From the previous theorem we obtain the following theorem.

Theorem 18. *Let (G, Σ) be a signed graph, let e be an even edge, and let u be an end of e . Then $[\{(1, 0)\} + \mathcal{I}^s(G/e, \Sigma)] \cup [\{0, 1\} + \mathcal{I}^s(G/e, \Sigma \Delta(\delta(u) \setminus \{e\})] \subseteq \mathcal{I}^s(G, \Sigma)$.*

Recall that a signed graph (H, Σ') is a *minor* of a signed graph (G, Σ) if (H, Σ') is sign-equivalent to a signed graph that can be obtained by contracting a sequence of even edges in a subgraph of (G, Σ) .

Corollary 19. *If (G', Σ') is a minor of (G, Σ) , then $\nu(G', \Sigma') \leq \nu(G, \Sigma)$.*

Proof. Let (G, Σ) have n vertices. As resigning on a subset $U \subseteq V(G)$ does not change $\nu(G, \Sigma)$, it suffices to prove this for the cases that (G', Σ') is obtained from (G, Σ) by deleting an edge or an isolated vertex, or contracting an even edge.

Suppose that (G', Σ') is obtained from (G, Σ) by deleting an isolated vertex. Then (G', Σ') has $n - 1$ vertices. Since $(n - 1 - \nu(G', \Sigma'), 0) \in \mathcal{I}^s(G', \Sigma')$, we obtain by Theorem 15 that $\{(n - 1 - \nu(G', \Sigma'), 0)\} + \mathbb{N}_1^2 \subseteq \mathcal{I}^s(G, \Sigma)$. Hence $(n - \nu(G', \Sigma'), 0) \in \mathcal{I}^s(G, \Sigma)$ and so $\nu(G', \Sigma') \leq \nu(G, \Sigma)$.

Suppose next that (G', Σ') is obtained from (G, Σ) by deleting an edge. Then (G', Σ') has n vertices. Since $(n - \nu(G', \Sigma'), 0) \in \mathcal{I}^s(G', \Sigma')$, we obtain by Theorem 15 that $(n - \nu(G', \Sigma'), 0) \in \mathcal{I}^s(G, \Sigma)$. Hence $\nu(G', \Sigma') \leq \nu(G, \Sigma)$.

Finally, suppose that (G', Σ') is obtained from (G, Σ) by contracting an even edge. Then (G', Σ') has $n - 1$ vertices. Since $(n - 1 - \nu(G', \Sigma'), 0) \in \mathcal{I}^s(G', \Sigma')$, we obtain by Theorem 17 that $\{(n - 1 - \nu(G', \Sigma'), 0)\} + \{(1, 0)\} \in \mathcal{I}^s(G, \Sigma)$. Hence $(n - \nu(G', \Sigma'), 0) \in \mathcal{I}^s(G, \Sigma)$ and so $\nu(G', \Sigma') \leq \nu(G, \Sigma)$. \square

In the same way one can prove the following corollary.

Corollary 20. *If $H = (W, F)$ is a minor of $G = (V, E)$ and (G, Σ) is a signed graph, then $\xi(H, \Sigma \cap F) \leq \xi(G, \Sigma)$.*

Corollary 21. [5] *If H is a minor of G , then $\mu(H) \leq \mu(G)$.*

Proof. It suffices to prove this for the cases that H is obtained from G by deleting an edge or an isolated vertex, or contracting an edge. Let G have n vertices.

Suppose that H is obtained from G by deleting an isolated vertex. Then H has $n - 1$ vertices. Since $(n - 1 - \mu(H) - 1, 1) \in \mathcal{I}^s(H, \emptyset)$, we obtain by Theorem 15 that $\{(n - 2 - \mu(H), 1)\} + \mathbb{N}_1^2 \subseteq \mathcal{I}^s(G, \emptyset)$. Hence $(n - 1 - \mu(H), 1) \in \mathcal{I}^s(G, \emptyset)$ and so $\mu(H) \leq \mu(G)$.

Suppose next that H is obtained from G by deleting an edge. Then H has n vertices. Since $(n - \mu(H) - 1, 1) \in \mathcal{I}^s(H, \emptyset)$, we obtain by Theorem 15 that $(n - 1 - \mu(H), 1) \in \mathcal{I}^s(G, \emptyset)$. Hence $\mu(H) \leq \mu(G)$.

Finally, suppose that H is obtained from G by contracting an edge. Then H has $n - 1$ vertices. Since $(n - 1 - \mu(H) - 1, 1) \in \mathcal{I}^s(H, \emptyset)$, we obtain by Theorem 17 that $\{(n - 2 - \mu(H), 1)\} + \{(1, 0)\} \in \mathcal{I}^s(G, \emptyset)$. Hence $(n - 1 - \mu(H), 1) \in \mathcal{I}^s(G, \emptyset)$ and so $\mu(H) \leq \mu(G)$. \square

Corollary 22. *Let H be a minor of G and let $k = |G| - |H|$. Then $\mathcal{I}^s(H) + \mathbb{N}_k^2 \subseteq \mathcal{I}^s(G)$.*

Proof. It suffices to prove this for the cases that H is a subgraph of G or that H is obtained from G by contracting an edge. Let $(p, q) \in \mathcal{I}^s(H)$. Then there exists a subset $\Sigma \subseteq E(H)$ such that $(p, q) \in \mathcal{I}^s(H, \Sigma)$. If H is a subgraph of G , then, by Theorem 15, $\{(p, q)\} + \mathbb{N}_k^2 \subseteq \mathcal{I}^s(G, \Sigma) \subseteq \mathcal{I}^s(G)$. If H is obtained from G by contracting an edge, then, by Theorem 17, $\{(p, q)\} + \mathbb{N}_1^2 \subseteq \mathcal{I}^s(G, \Sigma) \subseteq \mathcal{I}^s(G)$. \square

Corollary 23. [6] *If H is a minor of G , then $\nu(H) \leq \nu(G)$.*

Proof. There exists a subset $\Sigma \subseteq E(H)$ such that $\nu(H) = \nu(H, \Sigma)$. Since (H, Σ) is a minor of (G, Σ) , $\nu(H, \Sigma) \leq \nu(G, \Sigma)$. As $\nu(G, \Sigma) \leq \nu(G)$, we obtain that $\nu(H) \leq \nu(G)$. \square

In the same way one can prove the following corollary.

Corollary 24. [1] *If H is a minor of G , then $\xi(H) \leq \xi(G)$.*

5 The stable inertia sets of some other graphs and signed graphs

In this section we determine the inertia sets of some more graphs and signed graphs. We first determine the stable inertia sets of trees. Since for any tree T and any $\Sigma \subseteq E(T)$, the signed graph (T, Σ) is sign-equivalent to (T, \emptyset) , these results immediately give the stable inertia set of signed graphs (T, Σ) , where T is a tree and $\Sigma \subseteq E(T)$.

Proposition 25. *If $n \geq 2$, then $\mathcal{I}^s(P_n) = \mathcal{I}(P_n) = \mathbb{N}_{[n-1, n]}^2$, where P_n is a path on n vertices.*

Proof. By Lemma 16, $\mathbb{N}_{[n-1, n]}^2 \subseteq \mathcal{I}^s(P_n) \subseteq \mathcal{I}(P_n)$. As $\text{mr}(P_n) = n - 1$ (see [8, 9]), $\mathcal{I}(P_n) \subseteq \mathbb{N}_{[n-1, n]}^2$. Hence $\mathcal{I}(P_n) = \mathcal{I}^s(P_n) = \mathbb{N}_{[n-1, n]}^2$. \square

Proposition 26. $\mathcal{I}^s(K_{1,3}) = \mathcal{I}(K_{1,3}) = \mathbb{N}_{[3,4]}^2 \cup \{(1, 1)\}$.

Proof. By Lemma 16, $\mathbb{N}_{[3,4]}^2 \subseteq \mathcal{I}^s(K_{1,3}) \subseteq \mathcal{I}(K_{1,3})$. Since $\mu(K_{1,3}) = 2$, $(1, 1) \in \mathcal{I}^s(K_{1,3}, \emptyset)$, and hence $(1, 1) \in \mathcal{I}^s(K_{1,3})$. Since $\text{mr}(K_{1,3}) = 2$, $\mathcal{I}(K_{1,3}) \subseteq \mathbb{N}_{[2,4]}^2$. Since $\text{mr}_+(K_{1,3}) = 3$, $\mathcal{I}(K_{1,3}) \subseteq \mathbb{N}_{[3,4]}^2 \cup \{(1, 1)\}$. Hence $\mathcal{I}^s(K_{1,3}) = \mathcal{I}(K_{1,3}) = \mathbb{N}_{[3,4]}^2 \cup \{(1, 1)\}$. \square

Proposition 27. *If T is a tree with n vertices which has at least one vertex of degree ≥ 3 , then $\mathcal{I}^s(T) = (\mathbb{N}_{[3,4]}^2 \cup \{(1, 1)\}) + \mathbb{N}_{n-4}^2$.*

Proof. Since $K_{1,3}$ is a minor of T , $(\mathbb{N}_{[3,4]}^2 \cup \{(1, 1)\}) + \mathbb{N}_{n-4}^2 \subseteq \mathcal{I}^s(T)$ by Theorem 22. Since $\xi(T) \leq 2$ (see [1]), $\mathcal{I}^s(T) \subseteq \mathbb{N}_{[n-2, n]}^2$. Since $M_+(T) = 1$, $\mathcal{I}_s(T) = (\mathbb{N}_{[3,4]}^2 \cup \{(1, 1)\}) + \mathbb{N}_{n-4}^2$. \square

In general, $\mathcal{I}(T) \neq \mathcal{I}^s(T)$ for a tree T . For example, if $T = K_{1,4}$, then $M(K_{1,4}) = 3$, while $\xi(K_{1,4}) = 2$, so $\mathcal{I}^s(K_{1,4})$ is a strict subset of $\mathcal{I}(K_{1,4})$.

Theorem 28. *Let (G, Σ) be a signed graph and let v be a vertex of degree one. Let w be the vertex adjacent to v . Let (G_1, Σ_1) be obtained from (G, Σ) by deleting v , and let (H, Ω) be obtained from (G, Σ) by deleting v and w . If $\mathcal{I}^s(H, \Omega) = \mathcal{I}(H, \Omega)$, then*

$$\mathcal{I}^s(G, \Sigma) = \mathcal{I}^s(G_1, \Sigma_1) + \mathbb{N}_1^2.$$

Furthermore, if $\mathcal{I}(G_1, \Sigma_1) = \mathcal{I}^s(G_1, \Sigma_1)$, then $\mathcal{I}(G, \Sigma) = \mathcal{I}^s(G, \Sigma)$.

Proof. By Theorem 18,

$$\mathcal{I}^s(G_1, \Sigma_1) + \mathbb{N}_1^2 \subseteq \mathcal{I}^s(G, \Sigma) \subseteq \mathcal{I}(G, \Sigma).$$

(As edge e , we can use the edge connecting v and w .)

For the converse inclusion, suppose

$$(p, q) \in \mathcal{I}^s(G, \Sigma) \setminus (\mathcal{I}^s(G_1, \Sigma_1) + \mathbb{N}_1^2).$$

We may assume that $V(G) = \{1, \dots, n\}$ and that $v = n$ and $w = n - 1$. Let $R = V(G) \setminus \{v, w\}$. We claim that any matrix $A = [a_{i,j}] \in S(G, \Sigma)$ with p positive and q negative eigenvalues, and the SAP has $a_{v,v} = 0$. To see this, suppose for a contradiction that $a_{v,v} \neq 0$. If $a_{v,v} > 0$, then

$$B = A/A[v] = \begin{bmatrix} A[R] & A[R, w] \\ A[w, R] & a_{w,w} - a_{w,v}a_{v,v}^{-1}a_{v,w} \end{bmatrix} \in S(G_1, \Sigma_1)$$

has $p - 1$ positive and q negative eigenvalues. Suppose B does not have the SAP. Then there exists a nonzero symmetric matrix $X = [x_{i,j}]$ with $x_{i,j} = 0$ if i and j are adjacent in (G_1, Σ) or if $i = j$, such that $BX = 0$. Let

$$Y = [y_{i,j}] = \begin{bmatrix} X[R] & X[R, w] & -a_{v,v}^{-1}a_{v,w}X[R, w] \\ X[w, R] & 0 & 0 \\ -a_{v,v}^{-1}a_{v,w}X[w, R] & 0 & 0 \end{bmatrix}.$$

Then $y_{i,j} = 0$ if i and j are adjacent in (G, Σ) or if $i = j$, and $AY = 0$. Hence A would not satisfy the SAP. This contradiction shows that B has the SAP. Hence $(p - 1, q) \in \mathcal{I}^s(G_1, \Sigma_1)$, and so $(p, q) \in \{(1, 0)\} + \mathcal{I}^s(G_1, \Sigma_1)$, which is a contradiction. If $a_{v,v} < 0$, then $B = A/A[v] \in S(G_1, \Sigma_1)$ has p positive and $q - 1$ negative eigenvalues. In the same way as above one can show that B has the SAP. Then $(p, q) \in \{(0, 1)\} + \mathcal{I}^s(G_1, \Sigma_1)$, a contradiction. These contradictions show that $a_{v,v} = 0$.

A calculation shows that $A/A[\{v, w\}] = A[R]$. Since $A[\{v, w\}]$ has one positive and one negative eigenvalue, $A[R] = A/A[\{v, w\}]$ has $p - 1$ positive and $q - 1$ negative eigenvalues. Since $A[R] \in S(H, \Omega)$, $(p - 1, q - 1) \in \mathcal{I}(H, \Omega) = \mathcal{I}^s(H, \Omega)$. By Theorem 17, $\mathcal{I}^s(H, \Omega) + \mathbb{N}_1^2 \subseteq \mathcal{I}^s(G_1, \Sigma_1)$, and so $(p, q - 1) \in \mathcal{I}^s(G_1, \Sigma_1)$. Hence $(p, q) \in \mathcal{I}^s(G_1, \Sigma_1) + \{(0, 1)\} \subseteq \mathcal{I}^s(G_1, \Sigma_1) + \mathbb{N}_1^2$, which is a contradiction.

Assume now furthermore that $\mathcal{I}(G_1, \Sigma_1) = \mathcal{I}^s(G_1, \Sigma_1)$. Suppose $(p, q) \in \mathcal{I}(G, \Sigma) \setminus (\mathcal{I}^s(G_1, \Sigma_1) + \mathbb{N}_1^2)$. Then any matrix $A = [a_{i,j}] \in S(G, \Sigma)$ with p positive and q negative eigenvalues has $a_{v,v} = 0$. The proof of this is similar as above. Then $(p - 1, q - 1) \in \mathcal{I}(H, \Omega) = \mathcal{I}^s(H, \Omega)$ and then in the same way as above $(p, q) \in \mathcal{I}^s(G_1, \Sigma_1) + \mathbb{N}_1^2$. \square

Theorem 29. Let (G, Σ) be a signed graph, let v be a vertex of degree two in G with two neighbors u and w , and suppose the edges incident to v are even. Let (G_1, Σ) and (G_2, Σ_2) be obtained from (G, Σ) by deleting v and adding between u and w respectively an even edge and an odd edge. Let (H, Ω) be obtained from (G, Σ) by deleting v and identifying u and w . If $\mathcal{I}^s(H, \Omega) = \mathcal{I}(H, \Omega)$, then

$$\mathcal{I}^s(G, \Sigma) = [\{(1, 0)\} + \mathcal{I}^s(G_1, \Sigma)] \cup [\{(0, 1)\} + \mathcal{I}^s(G_2, \Sigma_2)].$$

Furthermore, if $\mathcal{I}^s(G_1, \Sigma) = \mathcal{I}(G_1, \Sigma)$ and $\mathcal{I}^s(G_2, \Sigma_2) = \mathcal{I}(G_2, \Sigma_2)$, then $\mathcal{I}(G, \Sigma) = \mathcal{I}^s(G, \Sigma)$

Proof. By Theorem 18,

$$[\{(1, 0)\} + \mathcal{I}^s(G_1, \Sigma)] \cup [\{(0, 1)\} + \mathcal{I}^s(G_2, \Sigma_2)] \subseteq \mathcal{I}^s(G, \Sigma).$$

(As edge e , we can use the edge connecting v and w .)

For the converse inclusion, suppose $(p, q) \in \mathcal{I}^s(G, \Sigma)$ and

$$(p, q) \notin [\{(1, 0)\} + \mathcal{I}^s(G_1, \Sigma)] \cup [\{(0, 1)\} + \mathcal{I}^s(G_2, \Sigma_2)].$$

We may assume that $V(G) = \{1, \dots, n\}$ and that $v = n$, $u = n - 1$, and $w = n - 2$. Let $R = V(G) \setminus \{u, v, w\}$ and $S = \{u, w\}$. We claim that any matrix $A = [a_{i,j}] \in S(G, \Sigma)$ with p positive and q negative eigenvalues, and the SAP has $a_{v,v} = 0$. To see this, suppose for a contradiction that $a_{v,v} \neq 0$. If $a_{v,v} > 0$, then $B = A/A[v] \in S(G_1, \Sigma)$ has $p - 1$ positive and q negative eigenvalues. Suppose B does not have the SAP. Then there exists a nonzero symmetric matrix $X = [x_{i,j}]$ with $x_{i,j} = 0$ if i and j are adjacent in (G_1, Σ) or if $i = j$, such that $BX = 0$. Let

$$Y = [y_{i,j}] = \begin{bmatrix} X[R] & X[R, S] & -a_{v,v}^{-1}X[R, S]A[S, v] \\ X[S, R] & 0 & 0 \\ -a_{v,v}^{-1}A[v, S]X[S, R] & 0 & 0 \end{bmatrix}.$$

Then $y_{i,j} = 0$ if i and j are adjacent in (G, Σ) or if $i = j$, and $AY = 0$. Hence A would not satisfy the SAP. This contradiction shows that B has the SAP. Hence $(p, q) \in [\{(1, 0)\} + \mathcal{I}^s(G_1, \Sigma)]$, a contradiction. If $a_{v,v} < 0$, then $B = A/A[v] \in S(G_2, \Sigma_2)$ has p positive and $q - 1$ negative eigenvalues, and in the same way as above one can show that B has the SAP. Then $(p, q) \in [\{(0, 1)\} + \mathcal{I}^s(G_2, \Sigma_2)]$, a contradiction. These contradictions show that $a_{v,v} = 0$.

We claim that $A/A[\{v, u\}] \in S(H, \Omega)$. By simultaneously scaling the u th row and column, we may assume that $a_{v,u} = -1$. Similarly, by simultaneously scaling the w th row and column we may assume that $a_{v,w} = -1$. Let $c = a_{w,w} + a_{u,u} - 2a_{u,w}$. Then

$$A/A[\{v, u\}] = \begin{bmatrix} A[R] & A[R, u] - A[R, w] \\ A[u, R] - A[w, R] & c \end{bmatrix} \in S(H, \Omega).$$

Since $A[\{v, u\}]$ has one positive and one negative eigenvalue, $A/A[\{v, u\}]$ has $p - 1$ positive and $q - 1$ negative eigenvalues. Hence $(p - 1, q - 1) \in \mathcal{I}(H, \Omega) = \mathcal{I}^s(H, \Omega)$. By Theorem 17, $\{(1, 0)\} + \mathcal{I}^s(H, \Omega) \subseteq \mathcal{I}^s(G_2, \Sigma_2)$ and $\{(0, 1)\} + \mathcal{I}^s(H, \Omega) \subseteq \mathcal{I}^s(G_1, \Sigma)$, and so $(p, q - 1) \in \mathcal{I}^s(G_2, \Sigma_2)$ and $(p - 1, q) \in \mathcal{I}^s(G_1, \Sigma)$. Hence $(p, q) \in [\{(1, 0)\} + \mathcal{I}^s(G_1, \Sigma)] \cup [\{(0, 1)\} + \mathcal{I}^s(G_2, \Sigma_2)]$, which is a contradiction.

The proof that $\mathcal{I}(G, \Sigma) = \mathcal{I}^s(G, \Sigma)$ if $\mathcal{I}^s(G_1, \Sigma) = \mathcal{I}(G_1, \Sigma)$ and $\mathcal{I}^s(G_2, \Sigma_2) = \mathcal{I}(G_2, \Sigma_2)$ goes along the same lines. \square

By applying induction on k and using Propositions 4 and 8, and Theorem 29, one can prove the following proposition.

Proposition 30. *If k is an even positive integer, then*

$$\mathcal{I}^s(C_k^e) = \mathcal{I}(C_k^e) = \mathbb{N}_{[k-1, k]}^2 \cup \{(2p+1, k-3-2p) \mid p = 0, \dots, k/2-2\}$$

and

$$\mathcal{I}^s(C_k^o) = \mathcal{I}(C_k^o) = \mathbb{N}_{[k-1, k]}^2 \cup \{(2p, k-2-2p) \mid p = 0, \dots, k/2-1\}.$$

If k is an odd integer ≥ 3 , then

$$\mathcal{I}^s(C_k^e) = \mathcal{I}(C_k^e) = \mathbb{N}_{[k-1, k]}^2 \cup \{(2p, k-2-2p) \mid p = 0, \dots, (k-3)/2\}$$

and

$$\mathcal{I}^s(C_k^o) = \mathcal{I}(C_k^o) = \mathbb{N}_{[k-1, k]}^2 \cup \{(2p+1, k-3-2p) \mid p = 0, \dots, (k-3)/2\}.$$

Corollary 31. *If k is an integer ≥ 2 , then $\mathcal{I}^s(C_k) = \mathbb{N}_{[k-2, k]}^2$.*

Proposition 32. $\mathcal{I}^s(K_4^d) = \mathcal{I}(K_4^d) = \mathbb{N}_{[2, 4]}.$

Proof. We may assume that K_4^d contains one odd edge e . Since K_4^d contains an odd triangle which can be obtained by deleting one vertex, $\mathbb{N}_{[2, 4]} \subseteq \mathcal{I}^s(K_4^d)$. Suppose for a contradiction that there exists a matrix $A \in S(K_4^d)$ with nullity > 2 . Let $f = vw$ be an even edge adjacent to e , where v is incident to e . As A has nullity > 2 , there exists a nonzero vector $x \in \ker(A)$ with $x_v = x_w = 0$. The other components of x must be nonzero. Let a_v and a_w be the v th and w th row of A , respectively. From $a_w x = 0$ it follows that one component is negative and the other positive. From $a_v x = 0$ it follows that both are either negative or positive. This contradiction shows that A has nullity at most two. Hence $\mathcal{I}^s(K_4^d) = \mathcal{I}(K_4^d) = \mathbb{N}_{[2, 4]}$. \square

Proposition 33. $\mathcal{I}^s(K_{2,3}^o) = \mathcal{I}(K_{2,3}^o) = \mathbb{N}_{[3, 5]}$ and

$$\mathcal{I}^s(K_{2,3}^e) = \mathcal{I}(K_{2,3}^e) = [\{(1, 1)\} + \mathbb{N}_{[0, 3]}] \cup \{(0, 4), (0, 5), (4, 0), (5, 0)\}.$$

Proof. Since $K_{2,3}^e$ is sign-equivalent to $(K_{2,3}, E(K_{2,3}))$, $(p, q) \in \mathcal{I}^s(K_{2,3}^e)$ if and only if $(q, p) \in \mathcal{I}^s(K_{2,3}^e)$. A similar statement holds for $K_{2,3}^o$.

Since the signed graph $K_{2,3}^e$ is bipartite, $(3, 0) \notin \mathcal{I}(K_{2,3}^e)$ and $(4, 0) \in \mathcal{I}^s(K_{2,3}^e)$. Since $\mu(K_{2,3}) = 3$ (see [5]), $(1, 1) \in \mathcal{I}^s(K_{2,3}^e)$. Hence $\mathcal{I}(K_{2,3}^e) = \mathcal{I}^s(K_{2,3}^e) = (\{(1, 1)\} + \mathbb{N}_{[0, 3]}) \cup \{(0, 4), (0, 5), (4, 0), (5, 0)\}$.

We now prove that $\mathcal{I}(K_{2,3}^o) = \mathcal{I}^s(K_{2,3}^o) = \mathbb{N}_{[3, 5]}$. By resigning if necessary, we may assume that $K_{2,3}^o$ has exactly one odd edge e . Removing the end of e with degree two leaves an even 4-cycle, and removing another vertex with degree two leaves an odd 4-cycle. Since $\mathcal{I}^s(C_4^o) \cup \mathcal{I}^s(C_4^e) = \mathbb{N}_{[2, 4]}$, we obtain from Theorem 17 that $\mathbb{N}_{[3, 5]} \subseteq \mathcal{I}^s(K_{2,3}^o) \subseteq \mathcal{I}(K_{2,3}^o)$.

Suppose that there exists a $(p, q) \in \mathcal{I}(K_{2,3}^o)$ with $p + q \leq 2$. Then there exists a matrix $A \in S(K_{2,3}^o)$ with nullity at least three. Let v be the vertex of degree three which is incident to e . Let w be the other vertex of degree three, and let u_1, u_2, u_3 be the vertices of degree two, where we assume that u_1 is incident to e . Since A has

nullity at least three, there exists a nonzero vector $x \in \ker(A)$ with $x_v = x_{u_2} = 0$. For each vertex i of $K_{2,3}^o$, let a_i denote the i th row of A . From $a_{u_2}x = 0$, we obtain that $x_w = 0$. From $a_vx = 0$, we obtain that x_{u_1} and x_{u_3} must have the same sign. However, from $a_wx = 0$, we obtain that x_{u_1} and x_{u_3} must have different signs. This contradiction shows that $\mathcal{I}^s(K_{2,3}^o) = \mathcal{I}(K_{2,3}^o) = \mathbb{N}_{[3,5]}$. \square

Corollary 34. $\mathcal{I}^s(K_{2,3}) = \mathbb{N}_{[3,5]}^2 \cup \{(1, 1)\}$.

Proposition 35. $\mathcal{I}^s(K_{3,3}) = \mathcal{I}(K_{3,3}) = \{(1, 1)\} \cup \mathbb{N}_{[3,6]}^2$.

Proof. Since $K_{3,3}$ has a K_4 -minor, $\mathbb{N}_{[3,6]}^2 \subseteq \mathcal{I}_s(K_{3,3})$ by Corollary 22 and Corollary 9. Since $\xi(K_{3,3}) = 4$ and $\text{mr}_+(K_{3,3}) \geq 3$ (see [3]), $(1, 1) \in \mathcal{I}^s(K_{3,3})$. Hence $\{(1, 1)\} \cup \mathbb{N}_{[3,6]}^2 \subseteq \mathcal{I}^s(K_{3,3})$. Since $\text{mr}(K_{3,3}) = 2$, $\text{mr}_+(K_{3,3}) = 3$, and $\mathcal{I}^s(K_{3,3}) \subseteq \mathcal{I}(K_{3,3})$, $\mathcal{I}^s(K_{3,3}) = \mathcal{I}(K_{3,3}) = \{(1, 1)\} \cup \mathbb{N}_{[3,6]}^2$. \square

6 Disjoint unions and 1-sums

A *cut-vertex* of a graph is a vertex whose deletion increases the number of connected components. A connected graph G is *2-connected* if G has at least three vertices and no cut-vertex. A *block* of a graph is a maximal connected subgraph without a cut-vertex. Let G be a graph and let C be a block of G . The *thin out* of C in G is the graph obtained from C by adding a pendant edge to each cut vertex v of G contained in C . So the thin out of G is a subgraph of G . These definitions extend to signed graphs. In this section we show how the stable inertia set of a signed graph (G, Σ) can be determined from the stable inertia sets of the thin out of the 2-connected blocks in (G, Σ) . A similar statement holds for graphs.

The following lemma will be used in the proofs of the theorems of this section.

Lemma 36. *Let (G, Σ) be a signed graph and let $A \in S(G, \Sigma_1)$. Suppose S_1 and S_2 are nonempty disjoint subsets of $V(G)$ such that $A[S_1, S_2] = 0$. Let $R = V(G) \setminus (S_1 \cup S_2)$. If A has the SAP, then it is not possible that there are nonzero vectors y and z such that $A[S_1 \cup R, S_1]y = 0$ and $A[S_2 \cup R, S_2]z = 0$.*

Proof. We may assume that $S_1 = \{1, \dots, k\}$, $R = \{k+1, \dots, m\}$ and $S = \{m+1, \dots, n\}$. If $A[S_1 \cup R, S_1]y = 0$ and $A[S_2 \cup R, S_2]z = 0$, then the matrix

$$X = [x_{i,j}] = \begin{bmatrix} 0 & 0 & yz^T \\ 0 & 0 & 0 \\ zy^T & 0 & 0 \end{bmatrix}$$

is nonzero, satisfies $x_{i,j} = 0$ if i and j are adjacent in G or if $i = j$, and $AX = 0$, which is a contradiction. \square

The *disjoint union* of two signed graphs (G_1, Σ_1) and (G_2, Σ_2) with disjoint vertex sets is the signed graph $(G_1 \cup G_2, \Sigma_1 \cup \Sigma_2)$.

Theorem 37. *Suppose (G, Σ) is the disjoint union of (G_1, Σ_1) and (G_2, Σ_2) , and let $n_1 = |V_1|$ and $n_2 = |V_2|$. Then $\mathcal{I}^s(G, \Sigma) = [\mathcal{I}^s(G_1, \Sigma_1) + \mathbb{N}_{n_2}^2] \cup [\mathcal{I}^s(G_2, \Sigma_2) + \mathbb{N}_{n_1}^2]$.*

Proof. Since (G_1, Σ_1) and (G_2, Σ_2) are subgraphs of (G, Σ) , $\mathcal{I}^s(G_1, \Sigma_1) + \mathbb{N}_{n_2}^2 \subseteq \mathcal{I}^s(G, \Sigma)$ and $\mathcal{I}^s(G_2, \Sigma_2) + \mathbb{N}_{n_1}^2 \subseteq \mathcal{I}^s(G, \Sigma)$. Hence $[\mathcal{I}^s(G_1, \Sigma_1) + \mathbb{N}_{n_2}^2] \cup [\mathcal{I}^s(G_2, \Sigma_2) + \mathbb{N}_{n_1}^2] \subseteq \mathcal{I}^s(G, \Sigma)$.

To see the converse inclusion, let $(p, q) \in \mathcal{I}^s(G, \Sigma)$. Let $A \in \mathcal{S}(G, \Sigma)$ have the SAP, p positive and q negative eigenvalues. Since A has the SAP, $A[V(G_1)]$ or $A[V(G_2)]$ is nonsingular by Lemma 36. We may assume that $A[V(G_1)]$ is nonsingular, as the other case is similar.

Suppose that $A[V(G_2)]$ does not have the SAP. Then there exists a nonzero symmetric matrix $X = [x_{i,j}]$ with $x_{i,i} = 0$ for all $i \in V(G_2)$, $x_{i,j} = 0$ for all $ij \in E(G_2)$, and $A[V(G_2)]X = 0$. Let

$$Y = [y_{i,j}] = \begin{bmatrix} 0 & 0 \\ 0 & X \end{bmatrix}.$$

Then $y_{i,i} = 0$ for all $i \in V(G)$, $y_{i,j} = 0$ for all $ij \in E(G)$, and $AY = 0$. Hence A does not have the SAP, contradicting the assumption.

Thus, $(p, q) \in \mathcal{I}^s(G_2, \Sigma_2) + \mathbb{N}_{n_1}^2$. \square

From Theorem 37, we immediately obtain the following corollaries.

Corollary 38. *Let G be a disjoint union of G_1 and G_2 . Let $n_1 = |G_1|$ and $n_2 = |G_2|$. Then $\mathcal{I}^s(G) = [\mathcal{I}^s(G_1) + \mathbb{N}_{n_2}^2] \cup [\mathcal{I}^s(G_2) + \mathbb{N}_{n_1}^2]$.*

Corollary 39. *If (G, Σ) is the disjoint union of (G_1, Σ_1) and (G_2, Σ_2) , then $\nu(G, \Sigma) = \max\{\nu(G_1, \Sigma_1), \nu(G_2, \Sigma_2)\}$.*

Corollary 40. *If (G, Σ) is the disjoint union of (G_1, Σ_1) and (G_2, Σ_2) , then $\xi(G, \Sigma) = \max\{\xi(G_1, \Sigma_1), \xi(G_2, \Sigma_2)\}$.*

If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are subgraphs of G such that $G = G_1 \cup G_2$ and $V_1 \cap V_2 = \{v\}$, then G is called the 1-sum of G_1 and G_2 at v . Let (G, Σ) be a signed graph. If G is the 1-sum of G_1 and G_2 at v , then (G, Σ) is called the 1-sum of $(G_1, \Sigma \cap E_1)$ and $(G_2, \Sigma \cap E_2)$ at v .

For $(p_1, q_1), (p_2, q_2) \in \mathbb{N}^2$, define $(p_1, q_1) + (p_2, q_2) = (p_1 + p_2, q_1 + q_2)$.

Theorem 41. *Let (G, Σ) be a connected signed graph and suppose (G, Σ) is the 1-sum of (G_1, Σ_1) and (G_2, Σ_2) at v , with both $E(G_1)$ and $E(G_2)$ nonempty. Let n_1 and n_2 be the number of vertices of G_1 and G_2 , respectively. For $i = 1, 2$, let (H_i, Σ_i) be the signed graph obtained from K_2 and G_i by identifying a vertex of K_2 with v . Then $\mathcal{I}^s(G, \Sigma) = [\mathcal{I}^s(H_1, \Sigma_1) + \mathbb{N}_{n_2-2}^2] \cup [\mathcal{I}^s(H_2, \Sigma_2) + \mathbb{N}_{n_1-2}^2]$.*

Proof. By negating around vertices, we may assume that for $i = 1, 2$ at least one edge of G_i incident to v is even. Since (H_1, Σ_1) and (H_2, Σ_2) are subgraphs of (G, Σ) , $\mathcal{I}^s(H_1, \Sigma_1) + \mathbb{N}_{n_2-2}^2 \subseteq \mathcal{I}^s(G, \Sigma)$ and $\mathcal{I}^s(H_2, \Sigma_2) + \mathbb{N}_{n_1-2}^2 \subseteq \mathcal{I}^s(G, \Sigma)$. Hence $[\mathcal{I}^s(H_1, \Sigma_1) + \mathbb{N}_{n_2-2}^2] \cup [\mathcal{I}^s(H_2, \Sigma_2) + \mathbb{N}_{n_1-2}^2] \subseteq \mathcal{I}^s(G, \Sigma)$.

To see the converse inclusion, let $A \in \mathcal{S}(G, \Sigma)$ have the SAP. For $i = 1, 2$, let $V_i = V(G_i)$, $E_i = E(G_i)$, and $S_i = V_i \setminus \{v\}$.

Suppose $A[S_1]$ is nonsingular. We may write

$$A = \begin{bmatrix} A[S_1] & A[S_1, v] & 0 \\ A[v, S_1] & a_{v,v} & A[v, S_2] \\ 0 & A[S_2, v] & A[S_2] \end{bmatrix}.$$

Let

$$P = \begin{bmatrix} I & -A[S_1]^{-1}A[S_1, v] & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I \end{bmatrix}.$$

By Sylvester's Law of Inertia,

$$P^T A P = \begin{bmatrix} A[S_1] & 0 & 0 \\ 0 & a_{v,v} - A[v, S_1]A[S_1]^{-1}A[S_1, v] & A[v, S_2] \\ 0 & A[S_2, v] & A[S_2] \end{bmatrix}$$

has the same inertia as A . The matrix

$$B := \begin{bmatrix} a_{v,v} - A[v, S_1]A[S_1]^{-1}A[S_1, v] & A[v, S_2] \\ A[S_2, v] & A[S_2] \end{bmatrix}$$

belongs to $\mathcal{S}(G_2, \Sigma_2)$ and $\text{pin}(B) + \text{pin}(A[S_1]) = \text{pin}(A)$.

Suppose for a contradiction that B does not have the SAP. Then there exists a nonzero symmetric matrix $X = [x_{i,j}]$ with $x_{i,i} = 0$ for all $i \in V_2$, $x_{i,j} = 0$ for all $ij \in E_2$, and $BX = 0$. Let $Z = -A[S_1]^{-1}A[S_1, v]X[v, S_2]$ and

$$Y = [y_{i,j}] = \begin{bmatrix} 0 & 0 & Z \\ 0 & 0 & X[v, S_2] \\ Z^T & X[S_2, v] & X[S_2] \end{bmatrix}.$$

Then $y_{i,i} = 0$ for all $i \in V$, $y_{i,j} = 0$ for all $ij \in E$, and $AY = 0$. Hence A does not have the SAP, contradicting the assumption.

Hence $\text{pin}(A) \in \mathcal{I}^s(G_2, \Sigma_2) + \mathbb{N}_{n_1-1}^2$ and since G_2 is a subgraph of H_2 , $\text{pin}(A) \in \mathcal{I}^s(H_2, \Sigma_2) + \mathbb{N}_{n_1-2}$. The case where $A[S_2]$ is nonsingular can be done similarly.

Therefore we may assume that both $A[S_1]$ and $A[S_2]$ are singular. Since A has the SAP, it is, by Lemma 36, not possible that there are nonzero vectors y and z such that $A[V_1, S_1]y = 0$ and $A[V_2, S_2]z = 0$; say there is no nonzero vector y such that $A[V_1, S_1]y = 0$.

Since there is no nonzero y with $A[S_1 \cup \{v\}, S_1]y = 0$, $A[S_1]$ has nullity 1. Let $x \in \ker(A[S_1])$ be nonzero. Let $w \in S_1$ with $x_w \neq 0$. Let $Q = S_1 \setminus \{w\}$. Then $A[Q]$ is nonsingular, because if $A[Q]$ were singular another (independent) vector could be constructed in $\ker(A[S_1])$. We may write

$$A = \begin{bmatrix} A[Q] & A[Q, w] & A[Q, v] & 0 \\ A[w, Q] & a_{w,w} & a_{w,v} & 0 \\ A[v, Q] & a_{v,w} & a_{v,v} & A[v, S_2] \\ 0 & 0 & A[S_2, v] & A[S_2] \end{bmatrix}.$$

Let

$$P = \begin{bmatrix} I & -A[Q]^{-1}A[Q, \{w, v\}] & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Here I_2 denotes the 2×2 identity matrix. By Sylvester's law of Inertia

$$P^T A P = \begin{bmatrix} A[Q] & 0 & 0 & 0 \\ 0 & b_{w,w} & b_{w,v} & 0 \\ 0 & b_{v,w} & b_{v,v} & A[v, S_2] \\ 0 & 0 & A[S_2, v] & A[S_2] \end{bmatrix},$$

where

$$\begin{bmatrix} b_{w,w} & b_{w,v} \\ b_{v,w} & b_{v,v} \end{bmatrix} = A[\{v, w\}] - A[\{v, w\}, Q] A[Q]^{-1} A[Q, \{v, w\}],$$

has the same inertia as A . Let

$$B := \begin{bmatrix} A[\{v, w\}] - A[\{v, w\}, Q] A[Q]^{-1} A[Q, \{v, w\}] & A[\{v, w\}, S_2] \\ A[S_2, \{v, w\}] & A[S_2] \end{bmatrix}.$$

The matrix B satisfies $\text{pin}(B) + \text{pin}(A[Q]) = \text{pin}(A)$. Since $A[S_1]$ is singular, we know that $b_{w,w} = a_{w,w} - A[w, Q] A[Q]^{-1} A[Q, w] = 0$. Suppose $b_{v,w} = a_{v,w} - A[v, Q] A[Q]^{-1} A[Q, w] = 0$. Then the vector

$$a = \begin{bmatrix} -A[Q]^{-1} A[Q, w] \\ 1 \end{bmatrix}$$

belongs to $\ker(A[V_1, S_1])$, contradicting the assumption. Therefore $b_{v,w} \neq 0$, that is, $B \in \mathcal{S}(H_2, \Sigma_2)$.

Suppose for a contradiction that B does not have the SAP. Then there is a nonzero symmetric matrix $X = [x_{i,j}]$ such that $x_{i,j} = 0$ if $i = j$ or $ij \in E$, and $BX = 0$. Let

$$Z = -A[Q]^{-1} A[Q, \{v, w\}] X[\{v, w\}, S]$$

and

$$Y = [y_{i,j}] = \begin{bmatrix} 0 & 0 & Z \\ 0 & 0 & X[\{v, w\}, S] \\ Z^T & X[S_1, \{v, w\}] & X[S_1] \end{bmatrix}.$$

Then Y is nonzero, $y_{i,j} = 0$ if $i = j$ or $ij \in E$, and $AY = 0$. Hence A would not have the SAP if B did not. Hence $\text{pin}(A) \in \mathcal{I}^s(H_2, \Sigma_2) + \mathbb{N}_{n_1-2}$. \square

Corollary 42. *Let G be a connected graph and suppose G is the 1-sum of G_1 and G_2 at v , both containing at least one edge. Let n_1 and n_2 be the number of vertices of G_1 and G_2 , respectively. For $i = 1, 2$, let H_i be the graph obtained from K_2 and G_i by identifying a vertex of K_2 with v . Then $\mathcal{I}^s(G) = [\mathcal{I}^s(H_1) + \mathbb{N}_{n_2-2}^2] \cup [\mathcal{I}^s(H_2) + \mathbb{N}_{n_1-2}^2]$.*

We now show how the stable inertia set of a signed graph (G, Σ) can be determined from stable inertia set of the thin out of each 2-connected block in (G, Σ) . We do this by induction on the number of vertices in G . Suppose (G, Σ) is not the thin out of a 2-connected block. If (G, Σ) is a disjoint union of (G_1, Σ_1) and (G_2, Σ_2) , then we apply Theorem 37 on (G_1, Σ_1) and (G_2, Σ_2) . We may therefore assume that (G, Σ) is connected. If (G, Σ) has a 2-connected block, then we apply Theorem 41. If (G, Σ)

has no 2-connected block, then each block is either a single edge or a class of parallel edges. We may assume that if a block has parallel edges, then the block is equal to K_2^- . If (G, Σ) is a 1-sum of $(G_1, \Sigma \cap E(G_1))$ and $(G_2, \Sigma \cap E(G_2))$ with both G_1 and G_2 containing at least two edges, then we apply Theorem 41. Hence, we may assume that if (G, Σ) is a 1-sum of $(G_1, \Sigma \cap E(G_1))$ and $(G_2, \Sigma \cap E(G_2))$, then G_1 or G_2 contains at most one edge.

Suppose now that (G, Σ) has a block that is equal to K_2^- . Then the number, n , of vertices of G is at most four and $\mathcal{I}^s(K_2^-) + \mathbb{N}_{n-2}^2 = \mathbb{N}_{[n-2, n]}^2 \subseteq \mathcal{I}^s(G, \Sigma)$. If $(p, q) \in \mathcal{I}^s(G, \Sigma)$ and $(p, q) \notin \mathbb{N}_{[n-2, n]}^2$, then $\xi(G, \Sigma) \geq 3$. Hence there exists a matrix $A \in S(G, \Sigma)$ with $\text{nullity } A \geq 3$. Let u, v be the vertices of K_2^- . Since $\text{nullity } A \geq 3$, there exists a nonzero vector x with $x_u = x_v = 0$. However, this forces $x_w = 0$ for any other vertex w of (G, Σ) . This contradiction shows that $\mathbb{N}_{[n-2, n]}^2 = \mathcal{I}^s(G, \Sigma)$.

Suppose finally that (G, Σ) has no blocks that are equal to K_2^- . Then (G, Σ) is sign-equivalent to (T, \emptyset) , where T is a tree. We then use Proposition 27.

7 Some characterizations

A signed graph has no K_2^- -minor if and only if it has no odd cycles, that is, if it is bipartite. Since $\nu(K_2^-) = 2$, each signed graph (G, Σ) with $\nu(G, \Sigma) \leq 1$ has no K_2^- -minor, that is, is bipartite. In fact, K_2^- is the only obstruction to having $\nu(G, \Sigma) \leq 1$.

Theorem 43. *A signed graph (G, Σ) has $\nu(G, \Sigma) \leq 1$ if and only if (G, Σ) is bipartite.*

Proof. If $\nu(G, \Sigma) \leq 1$, then (G, Σ) has no K_2^- -minor, that is, (G, Σ) has no odd cycle. Hence (G, Σ) is bipartite.

For the converse, suppose (G, Σ) is bipartite and $\nu(G, \Sigma) \geq 2$. Then, by Corollary 39, there exists a component (H, Σ_1) of (G, Σ) with $\nu(H, \Sigma_1) \geq 2$. Hence there exists a positive semidefinite matrix $A \in S(H, \Sigma_1)$ with $\text{nullity}(A) \geq 2$. This contradicts Lemma 5. \square

From Theorem 43 we obtain the following corollary.

Corollary 44. *[6] A graph G has $\nu(G) \leq 1$ if and only if G is a forest.*

Proof. If G is a forest, then for every subset $\Sigma \subseteq E(G)$, (G, Σ) is bipartite. Hence $\nu(G, \Sigma) \leq 1$ for every subset $\Sigma \subseteq E(G)$. Since $\nu(G) = \max\{\nu(G, \Sigma) \mid \Sigma \subseteq E(G)\}$, we obtain that $\nu(G) \leq 1$.

For the converse, suppose $\nu(G) \leq 1$ and G has a cycle C . Let e be an edge of C . Then $\nu(G, \{e\}) \leq \nu(G) \leq 1$. Since $(G, \{e\})$ has an odd cycle, we obtain a contradiction. \square

Theorem 45. *A signed graph (G, Σ) has $M_+(G, \Sigma) \leq 1$ if and only if (G, Σ) is connected bipartite.*

Proof. If (G, Σ) has $M_+(G, \Sigma) \leq 1$, then (G, Σ) is connected. Furthermore, since $\nu(G, \Sigma) \leq M_+(G, \Sigma) \leq 1$, (G, Σ) is bipartite.

Conversely, if (G, Σ) is connected bipartite, then, by Lemma 5, each matrix $A \in S(G, \Sigma)$ has nullity(A) ≤ 1 . \square

From Theorem 45, we obtain the following corollary.

Corollary 46. *A graph G has $M_+(G) \leq 1$ if and only if G is a tree.*

We now characterize the signed graphs (G, Σ) for which $\xi(G, \Sigma) \leq 1$.

Theorem 47. *A signed graph (G, Σ) has $\xi(G, \Sigma) \leq 1$ if and only if (G, Σ) is sign-equivalent to the signed graph (H, \emptyset) , where H is a graph whose underlying simple graph is a disjoint union of paths.*

Proof. If $\xi(G, \Sigma) \leq 1$, then (G, Σ) has no K_2^- -minor, that is, (G, Σ) has no odd cycle. Hence (G, Σ) is bipartite. Furthermore, (G, Σ) has no $K_{1,3}$ -minor, that is, G has no vertices with more than three neighbors. Thus G is up to parallel edges a disjoint union of paths.

For the converse, suppose G is a disjoint union of paths and $\xi(G, \Sigma) \geq 2$. Then, by Corollary 40, there exists a component (H, Σ_1) of (G, Σ) with $\xi(H, \Sigma_1) \geq 2$. Since $M(H) \geq \xi(H, \Sigma_1)$, we obtain that $M(H) \geq 2$. Since $M(P) \leq 1$ for any path P , we obtain a contradiction. \square

Theorem 48. *A signed graph (G, Σ) has $M(G, \Sigma) \leq 1$ if and only if (G, Σ) is sign-equivalent to a signed graph (H, \emptyset) , where H is a graph whose underlying simple graph is a path.*

Proof. If (G, Σ) has $M(G, \Sigma) \leq 1$, then (G, Σ) is connected. Furthermore, (G, Σ) has no K_2^- -minor and no $K_{1,3}$ -minor, as $M(G, \Sigma) \geq \xi(G, \Sigma) \geq \xi(K_2^-) = 2$ and $M(G, \Sigma) \geq \xi(K_{1,3}) = 2$. Hence (G, Σ) is sign-equivalent to the signed graph (H, \emptyset) , where H is a graph whose underlying simple graph is a path.

For the converse, suppose (G, Σ) is signed-equivalent to a signed graph (H, \emptyset) , where H is a graph whose underlying simple graph is a path. Since $M(H) \leq 1$, we obtain that $M(H, \emptyset) \leq 1$. Hence $M(G, \Sigma) \leq 1$. \square

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